Non-commutative quantum field theories, renormalization and emergent gravity

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Los Alamos, 19 January 2012

OVERVIEW

- ***Introduction to non-commutative spaces and quantum field theories thereon***
- ***Properties and issues of NCQFTs; and a promising candidate for a renormalizable NC gauge field model***
- ***Introduce matrix models and emergent gravity***

MOTIVATION

- incompatibility between GR and QFT

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \langle T_{\mu\nu} \rangle$$

lhs: classical Einstein tensor, rhs: ev of an operator

- natural limit in experimental length resolution:
better length resolution requires higher energy,
energy required for resolution of the Planck length
has a Schwarzschild radius of the Planck length

$$\Delta x_\mu \simeq \lambda_p = \sqrt{\frac{G\hbar}{c^3}} \simeq 10^{-33}\mathrm{cm}$$

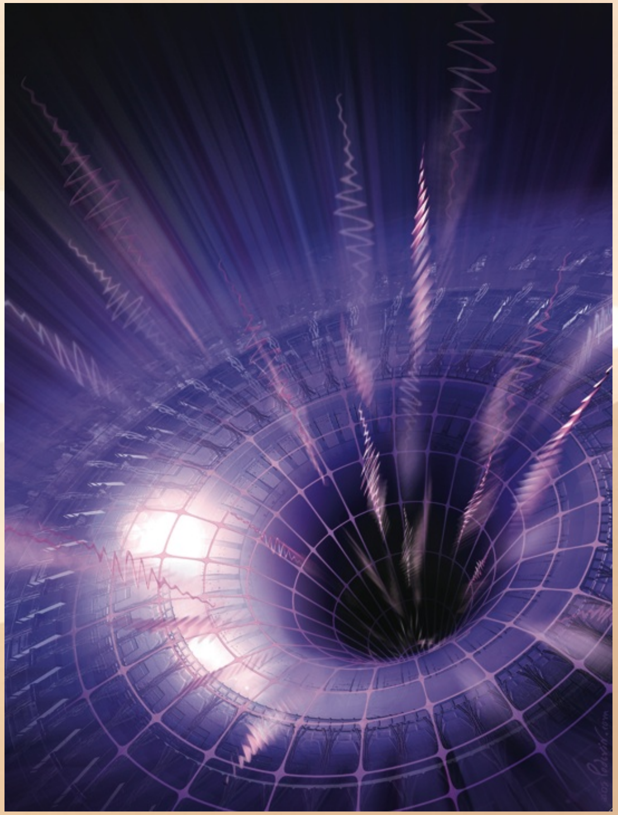


Image source:
http://web.physics.ucsb.edu/~giddings/sbgw/physics.html

Historically, the idea of a "minimal length" was initially introduced in order to smear out point-like interactions as UV regularization in QFTs (Snyder 1946).

Landau effect

Consider a charged particle in a constant magnetic field:

$$S = -m\int ds + e\int A_\mu dx^\mu \approx \frac{1}{2}\int dt\left(m\dot{\vec{x}}^2 + eB\epsilon_{ij}\dot{x}_i x_j\right)$$

Hamiltonian obtained by Legendre transformation:

$$H(\vec{x},\vec{p}) = \frac{1}{2m}(\vec{p}+e\vec{A})^2, \qquad \vec{p} = m\dot{\vec{x}} - e\vec{A}, \quad \vec{\pi} = m\dot{\vec{x}} = \vec{p} + e\vec{A}$$

$$[\hat{x}^j, \hat{p}_i] = i\delta_i^j \quad \Rightarrow \quad [\hat{\pi}_i, \hat{\pi}_j] = ieB\epsilon_{ij}$$

Energy spectrum is that of a harmonic oscillator: $E_n = \frac{eB}{m}\left(n+\frac{1}{2}\right)$ (Landau levels)

Spacial non-commutativity arises when $B \gg m \quad \Rightarrow \quad [\hat{x}^i, \hat{x}^j] = \frac{2i}{eB}\epsilon^{ij}$

Quantum Hall effect: quantized Hall resistance for low temp. and strong magn. field.
Steps correspond to the number of filled Landau levels.

WHAT IS A "NON-COMMUTATIVE" SPACE?

Geometric space $\Longleftrightarrow$ commutative C*-algebra (according to Gel'fand-Naimark theorem)

points — *pure states*

Non-commutative space $\Longleftrightarrow$ Non-commutative C*-algebra (A. Connes 1994)

Properties of a C*-algebra:

$$(a,b) \mapsto ab \in \mathcal{A} \qquad\qquad * : \mathcal{A} \mapsto \mathcal{A}\,,$$

$$a(b+c) = ab + ac\,, \qquad\qquad a^{**} = a\,, \qquad (ab)^* = b^* a^*\,,$$

$$(a+b)c = ac + bc\,, \quad \forall a,b,c \in \mathcal{A} \qquad\qquad (\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^*$$

$$\|a\| \geq 0\,, \qquad \|a\| = 0 \Leftrightarrow a = 0\,, \quad \|\alpha a\| = |\alpha| \|a\|\,,$$

$$\|a+b\| \leq \|a\| + \|b\|\,, \qquad \|ab\| \leq \|a\| \|b\|\,,$$

$$\|a^* a\| = \|a\|^2\,, \quad \forall a \in \mathcal{A}$$

commutator of the coordinates has the general form: $[\hat{x}^i, \hat{x}^j] = i\theta^{ij}(\hat{x})\,,$

$$\theta^{ij}(x) = \text{const}, \quad \theta^{ij}(x) = \lambda^{ij}_k \hat{x}^k, \quad i\theta^{ij}(x) = \left(\frac{1}{q}\hat{R}^{ij}_{kl} - \delta^i_l \delta^j_k\right)\hat{x}^k \hat{x}^l$$

Weyl quantization

assume non-commuting space-time coordinates:

$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}\,, \quad \Rightarrow$ leads to uncertainty relation $\quad \Delta x^\mu \Delta x^\nu \geq \frac{1}{2}|\theta^{\mu\nu}| \sim (\lambda_p)^2$

exists isomorphism mapping between NC algebra and commutative one, e.g. Weyl map

$$W : \mathcal{A} \to \widehat{\mathcal{A}}\,, \qquad x^i \mapsto \hat{x}^i$$

introduce a functions Weyl operator by

$\hat{\mathcal{W}}[f] := \int d^D x\, f(x)\hat{\Delta}(x)\,,$	$\hat{\Delta}(x) = \int \frac{d^D k}{(2\pi)^D} e^{ik_\mu \hat{x}^\mu} e^{-ik_\mu x^\mu}\,,$
$f(x) = \mathrm{Tr}\left(\hat{\mathcal{W}}[f]\hat{\Delta}(x)\right)\,,$	$\mathrm{Tr}\hat{\mathcal{W}}[f] = \int d^D x\, f(x)$

define derivation operator by $\quad [\hat{\partial}_\mu, \hat{x}^\nu] = \delta_\mu^\nu\,, \qquad [\hat{\partial}_\mu, \hat{\mathcal{W}}[f]] = \hat{\mathcal{W}}[\partial_\mu f]$

$$\boxed{\hat{\mathcal{W}}[f]\hat{\mathcal{W}}[g] = \hat{\mathcal{W}}[f \star g]}$$

Groenewold-Moyal space

definition of the Groenewold-Moyal *-product:

$$f(x) \star g(x) = \iint \frac{d^Dk \;\; d^Dk'}{(2\pi)^D(2\pi)^D} \tilde{f}(k)\tilde{g}(k') e^{-\frac{i}{2}k_\mu \theta^{\mu\nu} k'_\nu} e^{-i(k_\mu + k'_\mu)x^\mu}$$

$$= e^{\frac{i}{2}\theta^{\mu\nu}\partial^x_\mu \partial^y_\nu} f(x) g(y)\Big|_{x=y} \neq g(x) \star f(x)$$

... or with more fields:

$$f_1(x) \star \cdots \star f_m(x) = \iiint \frac{d^Dk_1}{(2\pi)^D} \cdots \frac{d^Dk_m}{(2\pi)^D} e^{i\sum\limits_{i=1}^{m} k_i x} \tilde{f}_1(k_1) \cdots \tilde{f}_m(k_m) e^{-\frac{i}{2}\sum\limits_{i<j}^{m} k_i \theta k_j}$$

invariance under cyclic permutations of the integral

$$\int d^Dx\, f(x) \star g(x) \star h(x) = \int d^Dx h(x) \star f(x) \star g(x)$$

and

$$\frac{\delta}{\delta f_1(y)} \int d^Dx \left(f_1 \star f_2 \star \cdots \star f_m\right)(x) = \left(f_2 \star \cdots \star f_m\right)(y)$$

QFT on deformed space-time

For a field theory in Euclidean space this means:
interaction vertices gain phases, whereas propagators remain unchanged, e.g.:

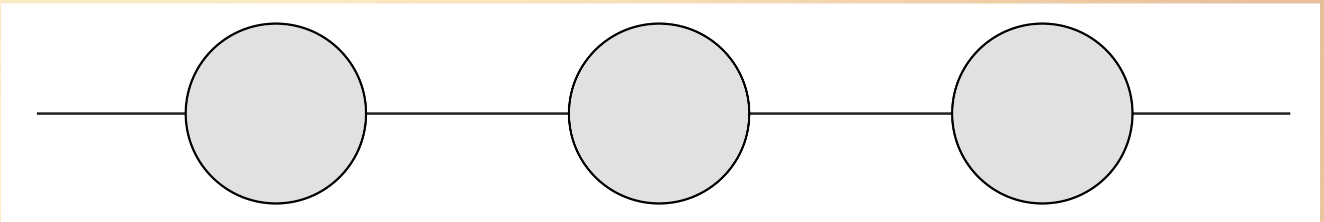
$$S = \mathrm{Tr}\left(\frac{1}{2}[\hat{\partial}_\mu, \hat{\mathcal{W}}[\phi]]^2 + \frac{m^2}{2}\hat{\mathcal{W}}[\phi]^2 + \frac{\lambda^2}{4!}\hat{\mathcal{W}}[\phi]^4\right)$$

$$= \int d^4x \left(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{m^2}{2}\phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi\right)$$

and some Feynman integrals (”non-planar diagrams”) have phases which act as UV-regulators

$$\frac{1}{4}\int d^4k \frac{e^{ik\theta p}}{k^2 + m^2} = \sqrt{\frac{m^2}{(\theta p)^2}} K_1\left(\sqrt{m^2(\theta p)^2}\right) \approx \frac{1}{(\theta p)^2} + c.m^2 \ln(\theta p)^2$$

⇨ ***origin of the UV/IR mixing problem***



Grosse-Wulkenhaar model

$$S = \int d^4x \left(\tfrac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \tfrac{m^2}{2}\phi^{\star 2} + 2\Omega^2 (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \tfrac{\lambda}{4!}\phi^{\star 4} \right), \qquad \tilde{x}_\mu := (\theta^{-1})_{\mu\nu} x^\nu$$

The propagator is known as the Mehler kernel and is the inverse of the operator

$$(-\Delta + 4\Omega^2 \tilde{x}^2 + m^2)$$

also observe that the action is "Langmann-Szabo" invariant, i.e.:

$$S[\phi; m, \lambda, \Omega] \mapsto \Omega^2 S[\phi; \frac{m}{\Omega}, \frac{\lambda}{\Omega^2}, \frac{1}{\Omega}]$$

and

$$[x^\mu \stackrel{\star}{,} f(x)] = i\theta^{\mu\nu}\partial_\nu f(x), \qquad \{x^\mu \stackrel{\star}{,} f(x)\} = 2x^\mu f(x)$$

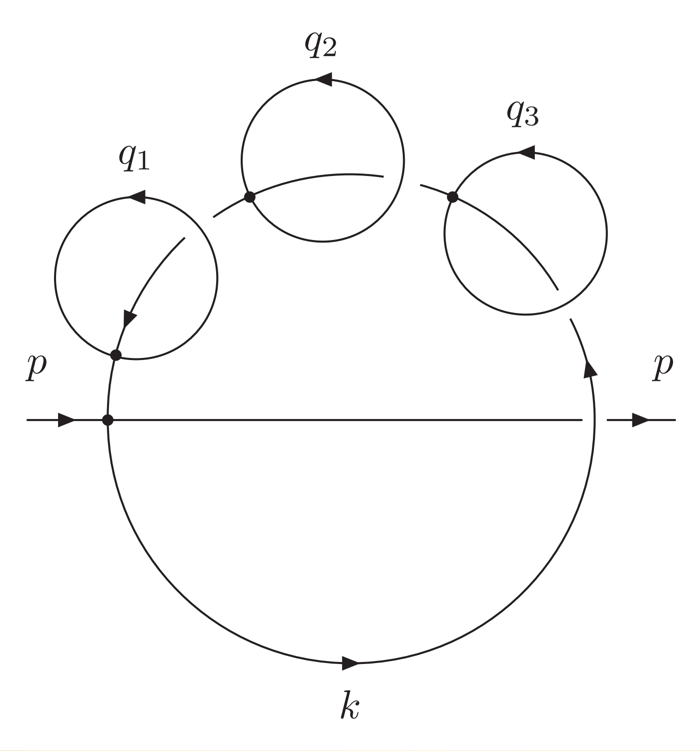
In 4-dimensional Euclidean space this model was proven to be renormalizable to all orders in perturbation theory.

Furthermore, there is no Landau ghost and the beta-function vanishes at the self-dual point.

A translation invariant alternative

$$S = \int d^4x \left(\tfrac{1}{2}\partial_\mu \phi \star \partial^\mu \phi + \tfrac{m^2}{2}\phi^{\star 2} - \phi(x) \star \frac{a^2}{\theta^2 \Box_x} \star \phi(x) + \tfrac{\lambda}{4!}\phi^{\star 4} \right)$$

propagator with infrared damping: $G(k) = \dfrac{1}{k^2 + m^2 + \frac{a^2}{(\theta k)^2}}$, $\lim_{k \to 0} G(k) = 0$



$$\Pi^{n \text{ np-ins.}}(p) \approx \lambda^2 \int d^4k \frac{e^{ik\theta p}}{((\theta k)^2)^n \left[k^2 + m^2 + \frac{a^2}{(\theta k)^2} \right]^{n+1}}$$

- $a = 0$: IR div. for $n \geq 2$, i.e. integrand $\sim (k^2)^{-n}$
- $a \neq 0$: finite, integrand $\sim \dfrac{1}{((\theta k)^2)^n \left[\frac{a^2}{(\theta k)^2}\right]^{n+1}} = \dfrac{(\theta k)^2}{(a^2)^{n+1}}$

This model by Gurau et. al., Commun.Math.Phys. **287** (2009) 275*, was proven to be renormalizable in 4-dimensional Euclidean space.*

Gauge fields on theta-deformed spaces

Star commutator of two Lie algebra valued functions:

$$[\alpha \stackrel{\star}{,} \beta] = \frac{1}{2}\{\alpha^a \stackrel{\star}{,} \beta^b\}[T^a, T^b] + \frac{1}{2}[\alpha^a \stackrel{\star}{,} \beta^b]\{T^a, T^b\}$$

$\Rightarrow$ *must always consider enveloping algebras, such as U(N), O(N) or USp(2N)*

Non-commutative Yang-Mills action:

$$\begin{aligned} S &= \frac{1}{4}\mathrm{Tr}\Big([\hat{\partial}_\mu, \hat{\mathcal{W}}[A]_\nu] - [\hat{\partial}_\nu, \hat{\mathcal{W}}[A]_\mu] - ig[\hat{\mathcal{W}}[A]_\mu, \hat{\mathcal{W}}[A]_\nu]\Big)^2 \\ &= \frac{1}{4}\int d^D x \, \mathrm{tr}_N \left(F_{\mu\nu}(x) \star F^{\mu\nu}(x)\right) \,, \end{aligned}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \underline{ig[A_\mu \stackrel{\star}{,} A_\nu]}$$

It is invariant under the infinitesimal gauge transformations

$$\begin{aligned} \delta_\alpha A_\mu(x) &= D_\mu \alpha(x) = \partial_\mu A(x) - ig[A_\mu(x) \stackrel{\star}{,} \alpha(x)] \,, \\ \delta_\alpha F_{\mu\nu}(x) &= -ig[F_{\mu\nu}(x) \stackrel{\star}{,} \alpha(x)] \end{aligned}$$

UV/IR mixing in NCGFTs

Need to add gauge fixing and ghost terms:

$$S = \frac{1}{4}\int d^4x \, \mathrm{tr}_N \big(F_{\mu\nu} \star F^{\mu\nu} + b \star \partial^\mu A_\mu + \tfrac{\xi}{2} b^{\star 2} - \bar{c} \star \partial^\mu D_\mu c\big)$$

This action is invariant under the BRST transformations

$$\begin{aligned} sA_\mu &= D_\mu c = \partial_\mu A - ig[A_\mu \stackrel{\star}{,} c]\,, \qquad & sc &= igc \star c\,, \\ s\bar{c} &= b\,, & sb &= 0\,, \\ s^2\varphi &= 0\,, \quad \forall\varphi \end{aligned}$$

IR divergent terms:

$$\Pi^{\mathrm{IR}}_{\mu\nu}(p) \propto \frac{\tilde{p}_\mu \tilde{p}_\nu}{(\tilde{p}^2)^2}\,, \qquad \tilde{p}^\mu := \theta^{\mu\nu} p_\nu\,,$$

$$\Gamma^{3A,\mathrm{IR}}_{\mu\nu\rho}(p_1, p_2, p_3) \propto \cos\left(\frac{p_1\theta p_2}{2}\right) \sum_{i=1,2,3} \frac{\tilde{p}_{i,\mu}\tilde{p}_{i,\nu}\tilde{p}_{i,\rho}}{(\tilde{p}_i^2)^2}$$

May try similar techniques as in the scalar case, but gauge symmetry makes matters more complicated...

Constructing an IR modified gauge field model

Example: implementing an IR damping similar to the scalar Gurau model.

$$\int d^4x\, \phi(x) \frac{a^2}{\theta^2 \Box} \phi(x) \quad \Rightarrow \quad \boxed{\int d^4x\, \frac{1}{4} F^{\mu\nu} \star \frac{a^2}{D^2 \widetilde{D}^2} \star F_{\mu\nu}}$$

$$\widetilde{D}_\mu = \theta_{\mu\nu} D^\nu\,, \qquad \delta_\alpha \left(\frac{1}{D^2} F \right) = ig[\alpha \stackrel{\star}{,} \frac{1}{D^2} F]$$

Drawback: infinite number of vertices ...

Proposition:
Use techniques known from the Gribov-Zwanziger action in QCD

$$\gamma^4 g^2 \int d^4x\, f^{abc} A^b_\mu (\mathcal{M}^{-1})^{ad} f^{dec} A^e_\mu \quad \Rightarrow \quad G^{ab}_{\mu\nu} = \frac{\delta^{ab}}{k^2 + \frac{\gamma^2}{k^2}} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right)$$

New gauge field action

$$S = \int d^4x \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + s\left((\bar{Q}_{\mu\nu\alpha\beta} B_{\mu\nu} + h.c.) \frac{1}{\widetilde{\square}} \left(f_{\alpha\beta} + \frac{\sigma\theta_{\alpha\beta}}{2} \tilde{f} \right) \right) \right.$$
$$\left. + s(\bar{\psi}^{\mu\nu} B_{\mu\nu}) + s \left(Q' \{ A_\mu \stackrel{\star}{,} A_\nu \} \frac{\tilde{\partial}_\mu \tilde{\partial}_\nu \tilde{\partial}_\rho}{\widetilde{\square}^2} A_\rho \right) + s(\bar{c} \partial_\mu A^\mu) \right]$$

$sA_\mu = D_\mu c\,,$	$sc = igc \star c\,,$	$s\bar{c} = b\,,$	$sb = 0\,,$
$s\,\bar{\psi}_{\mu\nu} = \bar{B}_{\mu\nu}\,,$	$s\,\bar{B}_{\mu\nu} = 0\,,$	$s\,B_{\mu\nu} = \psi_{\mu\nu}\,,$	$s\psi_{\mu\nu} = 0\,,$
$s\,\bar{Q}_{\mu\nu\alpha\beta} = \bar{J}_{\mu\nu\alpha\beta}\,,$	$s\,\bar{J}_{\mu\nu\alpha\beta} = 0\,,$		
$s\,Q_{\mu\nu\alpha\beta} = J_{\mu\nu\alpha\beta}\,,$	$s\,J_{\mu\nu\alpha\beta} = 0\,,$	$s\,Q' = J'\,,$	$s\,J' = 0\,.$

"Soft breaking" of the BRST symmetry like in the Gribov-Zwanziger case, leads to IR damping of the gauge field propagator, i.e. modifying the theory only in the infrared.

Soft breaking

$$\bar{Q}_{\mu\nu\alpha\beta}\big|_{\text{phys}} = Q_{\mu\nu\alpha\beta}\big|_{\text{phys}} = Q'\big|_{\text{phys}} = 0\,, \qquad J'\big|_{\text{phys}} = ig\gamma'^2\,,$$

$$\bar{J}_{\mu\nu\alpha\beta}\big|_{\text{phys}} = J_{\mu\nu\alpha\beta}\big|_{\text{phys}} = \frac{\gamma^2}{4}\left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right)$$

Propagator with IR damping

$$G^{AA}_{\mu\nu}(k) = \frac{1}{k^2\left(1 + \frac{\gamma^4}{((\theta k)^2)^2}\right)}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - f(\gamma,\sigma,k^2)\frac{(\theta k)_\mu(\theta k)_\nu}{(\theta k)^2}\right)$$

Is there a relation between the Gribov problem and UV/IR mixing in NCQFTs in the sense that solving one of the two automatically solves the other?

Can we derive some kind of “horizon condition” for the Gribov-like parameters?

How to prove renormalizability?

- Need a renormalization scheme that preserves gauge symmetry and works also in non-commutative space...

- The scalar model was proven using multiscale analysis which unfortunately breaks gauge symmetry in our case.

- Algebraic renormalization works well with models which have symmetries, but only if they are local in the quantum fields.
  Can we generalize the scheme to a non-commutative setting?

- Could expand for small deformation parameter theta (Seiberg-Witten map), but we would lose intrinsic properties of NCQFTs such as UV/IR mixing.

- At some point, need to also generalize to Minkowski space-time; this involves new time-ordering rules (some work in this direction has already been done).

Other non-commutative spaces

Fuzzy torus:

$$x^\mu \sim x^\mu + \Sigma_a^\mu$$

Fuzzy sphere:

$$[\hat{x}_i, \hat{x}_j] = i\frac{\theta}{r}\epsilon_{ijk}\hat{x}_k\,, \qquad r^2\mathbb{1} = (\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2)$$
$$[\hat{\partial}_i, \hat{\partial}_j] = \frac{1}{r}\epsilon_{ijk}\hat{\partial}_k\,, \qquad \hat{\partial}_i = -\frac{i}{\theta}\hat{x}_i$$

Kappa-deformed spaces:

$$[\hat{x}^\mu, \hat{x}^\nu] = i(a^\mu\delta_\sigma^\nu - a^\nu\delta_\sigma^\mu)\hat{x}^\sigma\,, \qquad \text{where e.g. } a^\mu = \kappa^{-1}\delta^{n\mu}$$

q-Deformation:

$$[\hat{x}^\mu, \hat{x}^\nu] = \left(\frac{1}{q}\hat{R}^{ij}_{kl} - \delta^i_l\delta^j_k\right)\hat{x}^k\hat{x}^l$$

Another approach involves a description of NC spaces in terms of Hopf algebras with deformed Leibnitz rules ("Twisted" gauge theories, NC generalizations to Einstein gravity, etc.)

Covariant coordinates & matrix models

star gauge transformation of a scalar field: $\phi(x) \to u(x) \star \phi(x) \star u(x)^\dagger$

In contrast to the commutative case, $x^\mu \star \phi(x)$ does not transform covariantly.

→ *define "covariant" coordinates:*

$$\boxed{\tilde{X}_\mu := \tilde{x}_\mu + gA_\mu}, \qquad \tilde{x}_\mu := \theta^{-1}_{\mu\nu} x^\nu$$

$$\tilde{X}_\mu \to u(x) \star \tilde{X}_\mu \star u(x)^\dagger \,,$$

$$\tilde{X}_\mu \phi(x) \to u(x) \star \tilde{X}_\mu \phi(x) \star u(x)^\dagger$$

$$\boxed{i \left[\tilde{X}_\mu \stackrel{\star}{,} \tilde{X}_\nu\right] = \theta^{-1}_{\mu\nu} - gF_{\mu\nu}}$$

NC Yang-Mills action: $\displaystyle \int d^4x F_{\mu\nu} \star F^{\mu\nu} \quad \to \quad -\frac{1}{g^2} \int d^4x \left[\tilde{X}_\mu \stackrel{\star}{,} \tilde{X}_\nu\right] \star \left[\tilde{X}^\mu \stackrel{\star}{,} \tilde{X}^\nu\right]$

Yang-Mills matrix model: $S_{YM} = -\mathrm{Tr}[X^a, X^b][X^c, X^d]\eta_{ac}\eta_{bd}$

Emergent gravity from matrix models

$$S_{YM} = -\mathrm{Tr}[X^a, X^b][X^c, X^d]\eta_{ac}\eta_{bd}$$

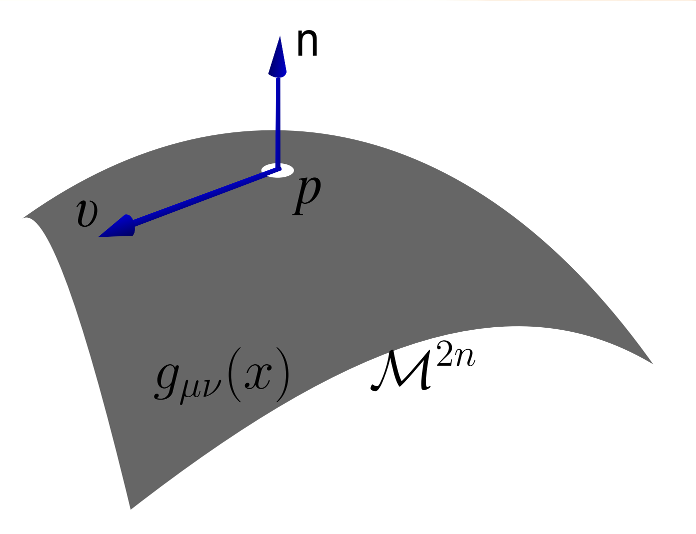
$$X^a = \left(X^\mu, \Phi^i\right), \ \mu = 1, \dots, 2n, \ i = 1, \dots, D - 2n,$$

so that $\Phi^i(X) \sim \phi^i(x)$ define embedding $\mathcal{M}^{2n} \hookrightarrow \mathbf{R}^D$

$g_{\mu\nu}(x) = \partial_\mu x^a \partial_\nu x^b \eta_{ab}$ (in semi-classical limit)

$$G^{\mu\nu} = e^{-\sigma}\theta^{\mu\rho}\theta^{\nu\sigma} g_{\rho\sigma} = -(\mathcal{J}^2)^\mu_\rho g^{\rho\nu}, \quad e^{-\sigma} \equiv \frac{\sqrt{\det \theta^{-1}_{\mu\nu}}}{\sqrt{\det G_{\rho\sigma}}}$$

for $2n = 4$: $\quad (\mathcal{J}^2)^\mu{}_\nu + \frac{(Gg)}{2}\delta^\mu{}_\nu + (\mathcal{J}^{-2})^\mu{}_\nu = 0$



$$\begin{aligned}
S[\phi] &= -\mathrm{Tr}\,[X^a, \phi]\,[X^c, \phi]\,\eta_{ac} \\
&\sim \int d^4x \sqrt{\det \theta^{-1}_{\mu\nu}}\, \{x^a, \phi\}_{\mathrm{PB}} \{x^c, \phi\}_{\mathrm{PB}} \eta_{ac} \\
&= \int d^4x \sqrt{\det \theta^{-1}_{\mu\nu}}\, \theta^{\mu\nu}\partial_\mu x^a \partial_\nu \phi\, \theta^{\rho\sigma}\partial_\rho x^c \partial_\sigma \phi\, \eta_{ac} \\
&= \int d^4x \sqrt{\det G_{\mu\nu}}\, G^{\nu\sigma}\partial_\nu\phi\partial_\sigma\phi
\end{aligned}$$

(cf. *Class.Quant.Grav.* **27** (2010) 133001)

Introducing the IKKT model

$$S_{\text{IKKT}} = \mathrm{Tr}\left(\left[X^a, X^b\right]\left[X_a, X_b\right] + \bar{\Psi}\not{D}\Psi\right)\ ,$$
$$\not{D}\Psi := \gamma_a\left[X^a, \Psi\right]\ , \qquad\qquad \{\gamma_a, \gamma_b\} = 2\eta_{ab}$$

IKKT matrix model is supersymmetric and expected to be renormalizable - cf. *Nucl.Phys.* **B498** (1997) 467.

Majorana-Weyl spinor $\Psi = \mathcal{C}\bar{\Psi}^T$, is invariant under SUSY:

$$\delta^1\Psi = \frac{i}{4}[X^a, X^b][\gamma_a, \gamma_b]\epsilon\,, \qquad \delta^1 X^a = i\bar{\epsilon}\gamma^a\Psi$$
$$\delta^2\Psi = \xi\,, \qquad \delta^2 X^a = 0$$

Further symmetries:

$$\begin{array}{llll} X^a \to U^{-1}X^aU\,, & \Psi \to U^{-1}\Psi U\,, & U \in U(\mathcal{H})\,, & \text{gauge inv.} \\ X^a \to \Lambda(g)^a_b X^b\,, & \Psi_\alpha \to \tilde{\pi}(g)^\beta_\alpha \Psi_\beta\,, & g \in \widetilde{SO}(D)\,, & \text{rotations,} \\ X^a \to X^a + c^a\mathbb{1}\,, & & c^a \in \mathbb{R}\,, & \text{translations} \end{array}$$

THE IKKT MODEL

$$S_{\mathrm{IKKT}} = \mathrm{Tr}\left(\left[X^a, X^b\right]\left[X_a, X_b\right] + \bar{\Psi}\gamma_a\left[X^a, \Psi\right]\right)$$

- Originally proposed as non-perturbative definition of type IIB string theory,
- Seems to provide a good candidate for quantum gravity and other fundamental interactions, i.e.

$$X^a = \begin{pmatrix} \bar{X}^\mu - \theta^{\mu\nu} A_\nu(\bar{X}^\mu) \\ \Phi^i(\bar{X}^\mu) \end{pmatrix}$$

- Here, we consider general NC brane configurations and their effective gravity in the matrix model,
- Assume soft breaking of SUSY below some scale Λ and compute the effective action using a Heatkernel expansion.

The fermionic action

$$S_\Psi = \mathrm{Tr}\Psi^\dagger \not{D}\Psi = \mathrm{Tr}\Psi^\dagger \gamma_a [X^a, \Psi]$$

$$e^{-\Gamma[X]} = \int d\Psi d\Psi^\dagger e^{-S_\Psi} = (\mathrm{const.}) \exp\left(\frac{1}{2}\mathrm{Tr}\log(\not{D}^2)\right)$$

$$\not{D}^2\Psi = \gamma_a\gamma_b[X^a, [X^b, \Psi]] = (\not{D}_0^2 + V)\Psi$$

- Consider fermions coupled to NC background
- Matrices X^a: perturbations around Moyal quantum plane

  $\Rightarrow$ introduce NC scale $\Lambda_{\mathrm{NC}}^4 = e^{-\sigma}$

$$[\bar{X}^\mu, \bar{X}^\nu] = i\bar{\theta}^{\mu\nu} = i\Lambda_{\mathrm{NC}}^{-2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$X^\mu = (\bar{X}^\mu + \mathcal{A}^\mu, \phi^i) = (\bar{X}^\mu - \bar{\theta}^{\mu\nu}A_\nu, \Lambda_{NC}^2\varphi^i)$$

Heatkernel expansion

$$\not{D}_0^2\Psi := \eta_{\mu\nu}[\bar{X}^\mu, [\bar{X}^\nu, \Psi]] = -\Lambda_{NC}^{-4}\bar{G}^{\mu\nu}\partial_\mu\partial_\nu\Psi$$

$$[X^\mu, X^\nu] = i(\bar{\theta}^{\mu\nu} + \mathcal{F}^{\mu\nu})\,, \qquad [X^\mu, \phi^i] = i\bar{\theta}^{\mu\nu}D_\nu\phi^i$$

$$\mathcal{F}^{\mu\nu} = -\bar{\theta}^{\mu\rho}\bar{\theta}^{\nu\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho - i[A_\rho, A_\sigma])\,,$$

$$D_\nu\phi = \partial_\nu\phi + i[A_\nu, \phi]$$

Consider a Duhamel expansion:

$$\frac{1}{2}\mathrm{Tr}\left(\log \not{D}^2 - \log \not{D}_0^2\right) \to -\frac{1}{2}\mathrm{Tr}\int_0^\infty \frac{d\alpha}{\alpha}\left(e^{-\alpha\not{D}^2} - e^{-\alpha\not{D}_0^2}\right)e^{-\frac{\Lambda_{\mathrm{NC}}^4}{\alpha\Lambda^2}}$$

$$= \Lambda^4\sum_{n\geq 0}\int d^4x\,\mathcal{O}\left(\frac{(p, A, \varphi)^n}{(\Lambda, \Lambda_{\mathrm{NC}})^n}\right)$$

Small parameters of expansion

In contrast to previous work, we consider a „semi-classical“ low energy regime characterized by

$$\epsilon(p) := p^2\Lambda^2/\Lambda_{NC}^4 \ll 1$$

Can expand UV/IR mixing terms as

$$e^{-p^2\Lambda_{\mathrm{NC}}^4/\alpha} \approx \sum_{m\geq 0} a_m \epsilon(p)^m$$

Avoids pathological phenomena which would appear if

$$\cancel{\Lambda \to \infty} \text{ and } \Lambda_{\mathrm{NC}} \text{ fixed}$$

Expansion in 3 small parameters:

$$\Gamma \sim \Lambda^4 \sum_{n,l,k\geq 0} \int d^4x \, \mathcal{O}\left(\epsilon(p)^n (\frac{p^2}{\Lambda_{\mathrm{NC}}^2})^l (\frac{p^2}{\Lambda^2})^k\right)$$

Effective NC gauge theory action

Weyl quantization map: $|p\rangle = e^{ip_\mu \bar{X}^\mu} \in \mathcal{A}$

$\bar{P}_\mu |p\rangle = ip_\mu |p\rangle\,, \qquad$ with $\bar{P}_\mu = -i\theta^{-1}_{\mu\nu}[\bar{X}^\nu, .]$

$\langle q|p\rangle = \mathrm{Tr}(|p\rangle\langle q|) = \mathrm{Tr}_{\mathcal{H}}(e^{-iq_\mu \bar{X}^\mu} e^{ip_\mu \bar{X}^\mu}) = (2\pi\Lambda_{\mathrm{NC}}^2)^2 \delta^4(p-q)$

$$\left[e^{ik\bar{X}}, e^{il\bar{X}}\right] = -2i\sin\left(\frac{k\bar{\theta}l}{2}\right) e^{i(k+l)\bar{X}}\,, \qquad \boxed{\not{D}_0^2 |p\rangle = \Lambda_{\mathrm{NC}}^{-4} \bar{G}^{\mu\nu} p_\mu p_\nu |p\rangle}$$

Can now compute the terms of the Duhamel expansion order by order:

$$\Gamma = \frac{1}{2}\int_0^\infty d\alpha\, \mathrm{Tr}(V e^{-\alpha \not{D}_0^2}) e^{-\frac{\Lambda_{\mathrm{NC}}^4}{\alpha\Lambda^2}}$$
$$- \frac{1}{4}\int_0^\infty d\alpha \int_0^\alpha dt'\, \mathrm{Tr}\Big(V e^{-t'\not{D}_0^2} V e^{-(\alpha-t')\not{D}_0^2}\Big) e^{-\frac{\Lambda_{\mathrm{NC}}^4}{\alpha\Lambda^2}} + \ldots$$

Gauge invariance of effective NCGFT

Adding up first 3 order contributions leads to the following order Λ^4 terms:

$$\Gamma_{\Lambda^4}(A,\varphi,p) = \frac{\Lambda^4 \mathrm{Tr}\mathbb{1}}{16\Lambda_{\mathrm{NC}}^4}\int \frac{d^4x}{(2\pi)^2}\sqrt{g}\Big(g^{\alpha\beta}D_\alpha\varphi^i D_\beta\varphi_i$$
$$- \frac{1}{2}\Lambda_{\mathrm{NC}}^4\big(\bar\theta^{\mu\nu}F_{\nu\mu}\bar\theta^{\rho\sigma}F_{\sigma\rho} + (\bar\theta^{\sigma\sigma'}F_{\sigma\sigma'})(F\bar\theta F\bar\theta)\big)$$
$$- 2\bar\theta^{\nu\mu}F_{\mu\alpha}g^{\alpha\beta}\partial_\nu\varphi^i\partial_\beta\varphi_i + \frac{1}{2}(\bar\theta^{\mu\nu}F_{\mu\nu})g^{\alpha\beta}\partial_\beta\varphi^i\partial_\alpha\varphi_i$$
$$+ \mathrm{h.o.}\Big)$$

These terms are manifestly gauge invariant

Free contribution:

$$\Gamma[\bar X] = -\frac{1}{2}\mathrm{Tr}\int_0^\infty \frac{d\alpha}{\alpha}e^{-\alpha \not{D}_0^2 - \frac{\Lambda_{\mathrm{NC}}^4}{\alpha\Lambda^2}} = -\frac{\Lambda^4\mathrm{Tr}\mathbb{1}}{8}\int\frac{d^4x}{(2\pi)^2}\sqrt{g}$$

Along with general geometrical considerations, this suffices to predict some loop computations

Effective matrix model action

consider $\Gamma_L[X] = \mathrm{Tr}\mathcal{L}\,(X^a/L), \quad L = \Lambda/\Lambda_{\mathrm{NC}}^2$

- Commutators correspond to derivative operators for gauge fields
- Leading term of eff. action can be written in terms of products of

$$J^a_b := i\Theta^{ac} g_{cb} = [X^a, X_b], \qquad \mathrm{Tr}J \equiv J^a_a = 0$$

Most general single-trace form of effective potential
+ input from free contribution to the effective action:

$$\Gamma_L[X] = \mathrm{Tr}V(X) + \mathrm{h.o.},$$
$$\mathrm{Tr}V(X) = -\frac{1}{4}\mathrm{Tr}\left(\frac{L^4}{\sqrt{-\mathrm{Tr}J^4 + \frac{1}{2}(\mathrm{Tr}J^2)^2}}\right) \sim -\frac{1}{8}\int \frac{d^4x}{(2\pi)^2}\Lambda^4(x)\sqrt{g}$$

SO(D) INVARIANCE OF GENERALIZED MM

Can reproduce gauge sector of induced result by a semi-classical analysis with vanishing embedding fields:

$$\frac{1}{\sqrt{\frac{1}{2}(\mathrm{Tr}J^2)^2 - \mathrm{Tr}J^4}}\bigg|_{\partial\phi^i=0} \sim \frac{\Lambda_{\mathrm{NC}}^4}{2}\bigg(1 + \frac{1}{2}\bar{\theta}^{\mu\nu}F_{\mu\nu} + \frac{1}{4}(\bar{\theta}F)^2 + \frac{1}{4}(\bar{\theta}F)(F\bar{\theta}F\bar{\theta}) + \mathcal{O}(F^4)\bigg)$$

Effective action can be written as a generalized matrix model with manifest SO(D) symmetry.

Can be further generalized to include curvature terms, e.g.:

$$\int \frac{d^4x}{(2\pi)^2}\sqrt{g}\,\Lambda(x)^2\Big(R + (\bar{\Lambda}_{\mathrm{NC}}^4 e^{-\sigma}\theta^{\mu\rho}\theta^{\eta\alpha}R_{\mu\rho\eta\alpha} - 4R) + c'\partial^\mu\sigma\partial_\mu\sigma\Big)$$

Such terms appear in the semiclassical limit of higher order matrix terms.

Conclusion

- Have discussed properties and problems (such as UV/IR mixing) of non-commutative quantum field theories, as well as renormalizable scalar models.
- Constructed a promising candidate for a renormalizable NC gauge field model, but need to prove renormalizability to all orders.
- Introduced matrix models, especially the IKKT model and its properties, such as emergent gravity.
- Computed the effective fermion action, first from NC field theory, then from the matrix model point of view.
- Many interesting open questions.

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Thank you for your attention!